# ON THE STABILITY OF EQUILIBRIUM STATES FOR SYSTEMS WITH DRY FRICTION

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Painlevé [1] in his book *Lectures on Friction* showed that the addition of dry friction forces does not alter the stability of equilibrium states of mechanical systems, for which the potential energy at the equilibrium state has an isolated minimum. In [2] and [3] a problem of stability of stationary motions of some particular mechanical systems with dry friction has been investigated: Some properties of the systems were determined such that the addtion of the dry friction forces does not affect the stability of the motion, provided that the systems are stable when subjected to the action of potential forces only. In [2] and [3], however, only such frictional forces are considered which appear during the sliding of the surfaces and which result in constant moments $\pm B$, independent from the normal forces (the forces are acting in the opposite direction to the relative sliding velocity). Such a model of dry friction, obviously, differs from the classical Coulomb model. It may, however, serve as a step toward the study of the problem in its classical formulation. The present paper is devoted tc such a study.

1. Consider a mechanical system subjected to stationary holonomic ideal constraints with the coordinates $q_1, \ldots, q_{n+k+l}$, and to nonholonomic scleronomous $(\partial A_{ij}/\partial t = 0)$ ideal constraints

$$A_{i1}\dot{q}_1 + \ldots + A_{i,n+k+l}\,\dot{q}_{n+k+l} = 0 \qquad (i = 1, \ldots, k)$$

with admissible displacements

$$A_{i1}\delta q_1 + \ldots + A_{i,n+k+l}\delta q_{n+l+k} = 0 \;. \qquad (i = 1, \ldots, k)$$

Let on the system be imposed some releasing constraints with dry friction

$$q_{n+1} \leqslant 0, \ldots, q_{n+l} \leqslant 0$$

If these inequalities reduce to equalities, then the bodies, or the points of the system, or external surfaces are sliding over other bodies of the system or over the external surfaces. Moreover, the frictional force remains proportional to the normal reaction $N_i > 0$ and is acting in the opposite direction to the sliding velocity ($N_i > 0$ if the bodies are pressing on each other).

Let us select at each contact point of one body three axes fixed on this body. Let the axis $z_i$ be directed along the external normal and the axes $x_i$ and $y_i$ complete a right-handed, orthogonal system of coordinate axes. Thus, the work of the reaction $N_i$ and the frictional force along the allowable displacements $\delta x_i$, $\delta y_i$, $\delta z_i$ of the points of the second body in this reference frame will be

$$N_i \delta z_i - k_i N_i \frac{v_{ix}}{|\mathbf{V}_i|} \delta x_i - k_i N_i \frac{v_{iy}}{|\mathbf{V}_i|} \delta y_i$$

where $v_{ix}$ and $v_{iy}$ are $x$- and $y$-components of the relative velocity vector $\mathbf{V}_i$ and $k_i > 0$ are friction coefficients.

In all subsequent considerations we shall not mention the fact that for the determination of the position of a system it is necessary to specify $n + l + k$ coordinates, although for the determination of the distribution of the velocities, it is necessary to specify only $n + l$ velocities. We shall also consider, without stating it explicitly, that all coordinates can be subjected to arbitrary excitations, although, perhaps, the length of the trajectory along which a system can be transferred from the initial state to a given excited state, remains bounded from below with the unlimited decrease of the amplitudes of some initial excitations. This can occur because of the existence of nonholonomic constraints. Such an assumption is necessary in view of the fact that we are not considering either the causes or the duration of the process of the build up of the excitation. We do not register therefore these assumptions in the indices, which simplifies the notation.

Let the allowable displacements $\delta x_i$, $\delta y_i$, $\delta z_i$ be expressed by the independent displacements $\delta q_1$, $\delta q_2$, $\ldots$, $\delta q_{n+l}$ as follows

$$\begin{aligned} \delta x_i &= \alpha_{i1}^1 \delta q_1 + \ldots + \alpha_{i,n+l}^1 \delta q_{n+l} \\ \delta y_i &= \alpha_{i1}^2 \delta q_1 + \ldots + \alpha_{i,n+l}^2 \delta q_{n+l} \\ \delta z_i &= \alpha_{i,n+1}^3 \delta q_{n+1} + \ldots + \alpha_{i,n+l}^3 \delta q_{n+l} \end{aligned}$$

where index $i$ varies over all contact points, and the last expressions do not contain $\delta q_1$, $\ldots$, $\delta q_n$ since all $\delta z_i = 0$ for

$$\delta q_{n+j} = q_{n+j} = \dot{q}_{n+j} = 0 \qquad (j = n+1, \ldots, n+l)$$

The velocities $v_{ix}$ and $v_{iy}$, being independent of time, are expressed in $\dot{q}_1, \ldots, \dot{q}_{n+l}$ by similar formulas:

$$\begin{aligned} v_{ix} &= \alpha_{i1}^1 \dot{q}_1 + \ldots + \alpha_{i,n+l}^1 \dot{q}_{n+l} \\ v_{iy} &= \alpha_{i1}^2 \dot{q}_1 + \ldots + \alpha_{i,n+l}^2 \dot{q}_{n+l} \\ z_{iz} &= \alpha_{i,n+1}^3 \dot{q}_{n+1} + \ldots + \alpha_{i,n+1}^3 \dot{q}_{n+l} \end{aligned}$$

Let us consider first the systems with a complete dissipation, i.e. for which the velocities $v_{ix}$ and $v_{iy}$ reduce to zero when $\dot{q}_1 = \ldots = \dot{q}_n = 0$.

Consider thus a system with the initial conditions $\dot{q}_1 = \dot{q}_{n+l} = 0$; $q_{n+1}^0 = \ldots = q_{n+l}^0 = 0$. The components of the frictional forces $R_{ix}$ and $R_{iy}$ in these initial conditions at equilibrium could be arbitrary, but such that their magnitudes $\sqrt{(R_{ix}^2 + R_{iy}^2)}$ should not exceed $k_i N_i$. The system remains in equilibrium if the sum of all virtual works of active forces $Q_i$ (independent of $t$ and continuous), of the frictional forces and positive normal reactions $N_i$ can be made, by a proper selection of $R_{ix}$ and $R_{iy}$, equal to zero for any virtual displacement of the system, ($R_{ix}$ and $R_{iy}$ must be the same for any virtual displacement). Thereby the following must be satisfied

$$Q_j' + \sum_i R_{ix} \alpha_{ij}^1 + R_{iy} \alpha_{ij}^2 + N_i \alpha_{ij}^3 = 0 \qquad (j = 1, \ldots, n+l) \tag{1.1}$$

where $Q_j'$ is the generalized force from the Appell equations [4], corresponding to the nonholonomic variable $\dot{q}_j$.

It may occur that in (1.1) there are more unknowns than the equations. In this case we shall impose on the solutions of these equations additional conditions which guarantee equilibrium.

1. Equations (1.1), (together with some additional hypothesis about the properties of $N_i$), can be satisfied only by the positive normal reactions.

2. For an arbitrary system of normal reactions $N_i > 0$ it is possible to determine a system $R_{ix}$, $R_{iy}$ such that all $R_{ix}$, $R_{iy}$ and $N_i$ together will satisfy (1.1), and $R_{ix}^2 + R_{iy}^2 < k_i^2 N_i^2$.

A natural additional hypothesis about the distribution of the normal forces may follow for instance, from the following considerations. A rigid body pressing on a rough horizontal flat surface in the form of a regular polygon exerts on this surface uniformly distributed force, directed down, if the mass of the body is symmetrically distributed about the vertical line passing through the geometrical center of the flat surface. Let now the additional mathematical hypothesis be expressed in the form of equalities or inequalities as follows

$$\begin{aligned} \Psi_s(q_i, \dot{q}_i, Q_i, N_i, R_{ix}, R_{iy}) > 0 \\ \varphi_s(q_i, \dot{q}_i, Q_i, N_i, R_{ix}, R_{iy}) = 0 \end{aligned} \qquad (s = 1, 2, \ldots p) \qquad (1.2)$$

where $\Psi_s$ and $\phi_s$ are continuous functions of their arguments, moreover, the arbitrary quantities $N_i$, $R_{ix}$ and $R_{iy}$ determined from the equalities in (1.2) are continuous functions of the remaining arguments of $\phi_s$. In the sequel we shall assume, without stating explicitely, that these relationships are attached to the equations of motion or equilibrium. If the initial conditions $q_1^0, \ldots, q_{n+l+k}^0$ are such that for arbitrary $N_i > 0$, it is possible to find $R_{ix}^2 + R_{iy}^2 < k_i^2 N_i^2$, then clearly, the same properties will obtain in some $\epsilon$-neighborhood of the initial conditions $q_i^0$

$$\sum_{i=1}^{n+k+l} (q_i^0 - q_i^{10})^2 = \sum_{i=1}^{n+k+l} x_i^2 \leqslant \varepsilon$$

Let the initial conditions be such that some, and not all, relative velocities are zero. Among the nonzero velocities there exist $\sigma$, and not more independent among themselves velocities, $v_1, \ldots, v_\sigma$. Let $v_1, \ldots, v_n$ represent a complete system of nonholonomic coordinates, independent for $q_1^1$, $q_{n+l}^0$; $\dot{q}_{n+1} = \dot{q}_{n+l} = 0$. Let now $S$ be the acceleration energy of the system depending on $l + n$ coordinates $v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+l}$, and their time derivatives, and let $Q_j$ be generalized force corresponding to these coordinates.

Let

$$\begin{aligned} v_{ix} &= \beta_{i1}^1 v_1 + \ldots + \beta_{i\sigma}^1 v_\sigma \\ v_{iy} &= \beta_{i1}^2 v_2 + \ldots + \beta_{i\sigma}^2 v_\sigma \\ v_{iz} &= \beta_{i,n+1}^3 \, v_{n+1} + \ldots + \beta_{i,n+l} v_{n+l} \end{aligned}$$

The equations of motion then are

$$\frac{\partial S}{\partial \dot{v}_j} = Q_j - \sum k_i N_i \frac{v_{ix}\beta_{ij}^1 + v_{iy}\beta_{ij}^2}{\sqrt{v_{ix}^2 + v_{iy}^2}} + \sum (R_{ix}\beta_{ij}^1 + R_{iy}\beta_{ij}^2 + N_i \beta_{ij}^3)$$
$$(j = 1, \ldots, n + l) \qquad (1.3)$$

where the first sum is extended over all contact points with nonvanishing $\mathbf{V}_i$; the second sum over all contact points with vanishing $\mathbf{V}_i$, and the third over all points. In what follows we shall always presuppose the above and we shall not indicate to the ranges of variation of the index $i$.

Solving (1.3) for $\dot{v}_{\sigma+1}, \ldots, \dot{v}_n, \ddot{q}_{n+1}, \ldots, \ddot{q}_{n+l}$ and equating the results to zero, we get

$$\sum_{k=1}^{n+l} \gamma_{kj} [ Q_j' - \delta_j - \sum k_i N_i \frac{v_{ix} \beta_{ij}^1 + v_{iy} \beta_{iy}^2}{\sqrt{v_{ix}^2 + v_{iy}^2}} + \\ + \sum (R_{ix}\beta_{ij}^1 + R_{iy}\beta_{ij}^2 + \sum N_i\beta_{ij}^3)] = 0 \qquad (j = \sigma + 1, \ldots, n+l) \qquad (1.4)$$

where $S_1 = \delta_1 \dot{v}_1 + \ldots + \delta_{n+1} \dot{v}_{n+l}$ is a part of $S$ linear relative to $\dot{v}_j$.

If the system (1.4) together with the hypothesis (1.2) could be satisfied only by a system $N_i$, $R_{ix}$ and $R_{iy}$ subjected to the inequalities $k_i N_i' > \sqrt{(R_{ix}^2 + R_{iy}^2)}$ then $\dot{v}_{\sigma+1}, \ldots, \dot{v}_{n+k+l}$ are zero and $\dot{v}_1, \ldots, v_\sigma$ are found from

$$\left(\frac{\partial S}{\partial \dot{v}_j}\right)^0 = Q_j + \sum k_i N_i \frac{v_{ix} \beta_{ij}^1 + v_{iy} \beta_{ij}^2}{\sqrt{v_{ix}^2 + v_{iy}^2}} \qquad (j = 1, \ldots, \sigma)$$

These quantities will be determined uniquely if and only if, all linear combinations $N_i > 0$ appearing on their right-hand side, are determined uniquely from (1.4) and (1.2).

A situation is more complicated if it is impossible to satisfy all inequalities

$$k_i N_i > \sqrt{R_{ix}^2 + R_{iy}^2}$$

In such cases the assumption $\dot{v}_{\sigma+1} = \ldots = \dot{v}_n = 0$ has to be dropped, and substituted by the assumption that some of the relative accelerations $\dot{v}_{\sigma+1}, \ldots, \dot{v}_\nu$ are different from zero and that in these points the frictional forces are directed opposite to the relative accelerations and equal to $k_i N_i$.

If it is possible to satisfy the equations

$$\frac{\partial S}{\partial \dot{v}_j} = Q_j' - \sum k_i N_i \frac{v_{ix} \beta_{ij}^1 + v_{iy} \beta_{ij}^2}{\sqrt{v_{ix}^2 + v_{iy}^2}} - \sum k_i N_i \frac{\dot{v}_{ix} \beta_{ij}^1 + \dot{v}_{ij} \beta_{ij}^2}{\sqrt{\dot{v}_{ix}^2 + \dot{v}_{iy}^2}} + \\ + \sum (\beta_{ij}^3 N_i + \sum R_{ix} \beta_{ij}^1 + R_{iy} \beta_{ij}^2) \qquad (j = 1, \ldots, n+l)$$

(where the second sum is extended over the points with nonvanishing $\dot{v}_{ix}$, $\dot{v}_{iy}$) it is possible to satisfy the accelerations $\dot{v}_{\nu+1} = \ldots = \dot{v}_n = \dot{v}_{n+1} = \ldots = \dot{v}_{n+l} = 0$, and also only the reactions

$$N_i > \frac{1}{k_i} \sqrt{R_{ix}^2 + R_{iy}^2}$$

(at the points with $\dot{v}_{ix} = \dot{v}_{iy} = 0$), then $\dot{v}_{\gamma+1} = \ldots = \dot{v}_n = 0$, and $\dot{v}_1$, $\ldots$, $\dot{v}_\nu$ obtained from these equations will constitute an admissible system [5]. It is interesting to note that for an arbitrary admissible system $\dot{v}_1, \ldots, \dot{v}_\nu$, the function

$$\Pi = -S_1 + \sum Q_i \dot{v}_i - \sum k_i N_i \sqrt{\dot{v}_{ix}^2 + \dot{v}_{iy}^2} - \sum k_i N_i \frac{\dot{v}_j (v_{ix} \beta_{ij}^1 + v_{iy} \beta_{ij}^2)}{\sqrt{v_{ix}^2 + v_{iy}^2}}$$

can have only positive values. For, multiplying (1.4) by $\dot{v}_j$ and adding, we obtain

$$2S_2 = \Pi \geqslant 0$$

where $S_2 > 0$, a part of $S$, is quadratic in $\dot{v}_i$. Function $\Pi$ can be interpreted as "work" of all active forces, frictional forces, and of a part of the inertial forces applied to the system on its actual acceleration. Another part of the inertia forces constitute inertia forces which would be applied to the system if it were acted upon by the active forces for which the accelerations $\dot{v}_i$ would vanish.

Let us now analyze all systems with initial conditions $q_i^0$, $v_i$ such that the absolute values of the velocities $v_i$ are sufficiently small and have the following properties:

a) For $q_i^0 = 0$, $v_i = 0$ the following inequalities are satisfied

$$k_i N_i > \sqrt{R_{ix}^2 + R_{iy}^2} - \delta'$$

where $N_i$ and $R_i$ are the solutions of (1.1).

b) For arbitrary small $|v_i| \neq 0$, $|x_i| \neq 0$, the new reactions $R'_{ix}$, $R'_{iy}$ and $N_i'$, which are the solutions of any of the variants of (1.3), satisfy

$$k_i N_i' \geqslant \sqrt{R'^2_{ix} + R'^2_{iy}} - \delta' \qquad (1.5)$$

$$k_i N_i' \geqslant \sqrt{R_{ix}^2 + R_{iy}^2} - \delta' \qquad (1.6)$$

where $\delta' > 0$ is a small constant quantity.

If it is impossible to determine either $N_i$, $N_i'$ or $R_i'$, $R_i$ uniquely, then we shall consider, that for an arbitrary system $N_i'$ it is possible to determine a system $R_{ix}$, $R_{iy}$ which will satisfy inequality (1.6). Mechanically this means that the normal reactions, at the beginning of the sliding with small velocity, vary from their static values continuously or make sufficiently small jumps.

The region of the initial values satisfying (a) and (b) above we shall call, following Bulgakov [6], the region of stagnation. We shall prove below a theorem, which will also justify the terminology used here.

*Theorem.* An arbitrary equilibrium state inside of the stagnation region is stable, and an arbitrary excited motion about this equilibrium state, possessing an arbitrary small kinetic energy, will cease after a finite period of time.

It is easy to note that each point of the stagnation region $q_i^0$, $\dot{q}_i^0$ is an interior point of the region, and thus, there exist a spherical neighborhood

$$\sum (q_i - q_i^0)^2 + \sum v_i^2 \leqslant R$$

of such point consisting entirely of the stagnation points of the region. For the left sides of any of the equations which serve to prove that a given point is a point of a stagnation region, depend in a continuous manner on $q_i$, $\dot{q}_i$, and the inequalities in the condition (a) and (b) contain some additional reserve $\delta'$.

Let us now consider any system of initial displacements and velocities

$$\sum (q_i' - q_i^0)^2 + \sum v_i^2 = \sum (x_i^2 + v_i^2) \leqslant \lambda > 0$$

For sufficiently small $\lambda$ this motion could be considered as being excited about its initial equilibrium state $q_i'$, and further discussion could be conducted considering the excitations of the velocities only.

From the theorem of the variation of the kinetic energy $T$ we have

$$\frac{d}{dt} T = \sum (Q_i^{0\prime} + \delta Q_i') \dot{v}_i - \sum k_i N_i' \sqrt{v_{ix}^2 + v_{iy}^2}$$

From (1.1) and (1.3) we obtain

$$\frac{d}{dt} T = \sum \left(R_{ix} v_{ix} + R_{iy} v_{iy} - k_i N_i' \sqrt{v_{ix}^2 + v_{iy}^2}\right) - \sum \delta Q_j' v_j$$

We shall prove now that

$$\sum \left(R_{ix} v_{ix} + R_{iy} v_{iy} - k_i N_i' \sqrt{v_{ix}^2 + v_{iy}^2}\right) - \sum \delta Q_j' v_j < - \theta \sqrt{T} \qquad (1.7)$$

in some region $\Sigma x_i^2 + v_i^2 < R_1 < R$, where $\theta > 0$ is some constant, which can be chosen independently from $R$, if only $|R_1|$ is sufficiently small.

Each term of (1.7) is negative in the region $R_1$ since

$$(R_{ix} + \delta Q_{ix})\, v_{ix} + (R_{iy} + \delta Q_{iy})\, v_{iy} < k_i N_i' \sqrt{v_{ix}^2 + v_{iy}^2}$$

Here $Q_{ix}$ is a generalized force corresponding the coordinate $v_{ix}$, etc. Squaring both sides, we obtain

$$[(R_{ix} + \delta Q_{ix})\, v_{ix} + (R_{iy} + \delta Q_{iy})\, v_{iy}]^2 - k_i N_i^2 (v_{ix}^2 + v_{iy}^2) < 0$$

For the Sylvester conditions for this quadratic form in $v_{ix}$ and $v_{iy}$, in sufficiently small $R_1$, reduces to $\sqrt{(R_{ix}{}^2 + R_{iy}{}^2)} \leqslant k_i N_i - \delta'$, i.e. to (1.6) since $\delta Q_{ix}$ is arbitrary small because of the continuity of $Q_j$.

Consider now lower limits of $k_i N_i$, denote them by $(k_i N_i)^0$, and let $\mu_i$ denote the upper limits of the moduli of $\delta Q_i$ in $R_1$. The function

$$\Phi = \sum_i R_{ix} v_{ix} + R_{iy} v_{iy} - (k_i N_i)^0 \sqrt{v_{ix}^2 + v_{iy}^2} + \Sigma \mu_j |v_j|$$

clearly exceeds the left side of (1.7). It is a homogeneous function of first order in $v_i$. Consequently, if $\gamma$ is its negative maximum on the sphere $R_1$, then everywhere in $R_1$

$$-\Phi > +\frac{\gamma}{\sqrt{R}} \sqrt{v_1^2 + \ldots + v_n^2}$$

Obviously, there exist always such $R > R_1$, that for arbitrary $\Sigma x_i^2 + v_i^2 \leqslant R$, the following inequality is valid

$$\sqrt{T} < \sigma' \sqrt{v_1^2 + \ldots + v_n^2}$$

moreover, $0 < \sigma = \text{const}$ is independent of $R_1$. Thus

$$-\Phi > \frac{\gamma}{\sigma' \sqrt{R}} \sqrt{T} = \theta \sqrt{T}$$

which completes the proof.

Hence

$$\frac{d}{dt} T < -\theta \sqrt{T} \quad \text{in the region} \quad \sum x_s^2 + v_s^2 \leqslant R_1$$

If $\lambda$ is sufficiently small then $T^\circ$ is arbitrarily small, therefore, for an arbitrary variant of the equations of motion

$$\sqrt{T} - \sqrt{T^\circ} \leqslant \theta/2\, (t - t_0)$$

It is seen from the last equality that the motion is asymptotically stable relative to the velocities, and it will necessarily cease after some interval of time bounded above by $L(\lambda) > 0$.

It is also easy to demonstrate, that there exist such a constant $l(\lambda) > 0$ that the moduli of all $v_i$ satisfy

$$|v_i| \leqslant \frac{l}{n} [\sqrt{T^\circ} - \theta/2 (t - t_0)]$$

From the last inequalities we obtain the following evaluation

$$\sum |x_i| \leqslant l' [\sqrt{T^\circ} (t - t_0) - \theta/4 (t - t_0)^2 + C]$$

where $l' > 0$, $C > 0$ are some constants. All these evaluations are valid as long as the motion is in progress, and it follows that the initial equilibrium is stable.

2. Consider now a system with a partial dissipation. Let $v_1, \ldots, v_\sigma$, $\sigma < n$ be independent among all $v_{ix}$, $v_{iy}$, and let the frictional forces do no work along the admissible displacements $\delta q_{\sigma+1}, \ldots, \delta q_n$. We also assume that the equations expressing nonholonomic constraints are linear in $v_1, \ldots, v_\sigma$.

As nonholonomic coordinates take

$$v_1, \ldots, v_\sigma, \dot{q}_{\sigma+1}, \ldots, \dot{q}_n, v_{n+1}, \ldots, v_{n+l}$$

Let the generalized forces corresponding to these coordinates in the Appell equations be $Q_1, \ldots, Q_{n+l}$. The equilibrium state

$$q^\circ_1, \ldots, q^\circ_{n+l+k} \tag{2.1}$$

will take place if the equations

$$Q_{\sigma+1} = \ldots = Q_n = 0, \ Q_j + \sum R_{ix}\beta_{ij}^1 + R_{iy}\beta_{ij}^2 + N_i\beta_{ij}^3 = 0$$

can be satisfied only for $k_i N_i > \sqrt{(R_{ix}^2 + R_{iy}^2)}$. Let also in any equations of motion, derived for sufficiently small variations of the coordinates and velocities, the normal reactions $N_i'$ and the tangential reactions $R'_{ix}$ and $R'_{iy}$ satisfy (1.5) and (1.6). This means that if at any arbitrary instant $t^*$ the expression $v_{ix}^2 + v_{iy}^2$ is reduced to zero, then it will remain zero for all $t > t^*$, until the motion will abandon some small neighborhood of the equilibrium state.

Let $Q_{\sigma+1}, \ldots, Q_n$ be dependent only on the coordinates and in the vicinity of (2.1) satisfy

$$\frac{\partial U}{\partial q_{\sigma+1}} = Q_{\sigma+1}, \ldots, \frac{\partial U}{\partial q_n} = Q_n$$

where $U$ is a holomorphic function of all coordinates in (2.1).

The coordinates $q_{\sigma+1}, \ldots, q_n$ we shall call free coordinates, and the remaining quasi-free and being inside of the stagnation region. This terminology facilitates the formulation of the theorem below.

*Theorem.* If $U_1$, the expansion of the function $U$ in powers of the variations of the free coordinates only, starts with a quadratic negative-definite form in respect to its variables, and the quasi-free coordinates are inside of the stagnation region, then such equilibrium is stable. Moreover, the excitation will reduce over a finite period of time to undamped small vibrations of the free coordinates only. Besides, this motion will be such as though some additional ideal constraints were imposed on the system expressing the constancy of the non-free coordinates when the latter differ very little from their values at the equilibrium state. Indeed,

$$\sum_{i=\sigma+1}^{n} Q_i\dot{q}_i = \sum_{i=\sigma+1}^{n} \frac{\partial U}{\partial q_i}\dot{q}_i = \frac{dU}{dt} - \sum_{\substack{i\leqslant\sigma\\ i>n}} \frac{\partial U}{\partial q_i}\dot{q}_i$$

Consider

$$\frac{dM}{dt} = \frac{d}{dt}\Big[T - U + \sum_{\substack{i\leqslant\sigma\\ i>n}} \left(\frac{\partial U}{\partial q_i}\right)^{\circ} x_i + \sum_{\substack{i\leqslant\sigma\\ i>n}} \beta_i x_i^2\Big] =$$

$$= \sum_{i=\sigma+1}^{n} Q_i\dot{q}_i + \sum_{i=1}^{\sigma} Q_i v_i - \sum k_i N_i \sqrt{v_{ix}^2 + v_{iy}^2} - \frac{dU}{dt} +$$

$$+ \sum_{\substack{i\leqslant\sigma\\ i>n}} \left(\frac{\partial U}{\partial q_i}\right)\dot{x}_i + 2\beta \sum_{\substack{i\leqslant\sigma\\ i>n}} x_i\dot{x}_i$$

where $(\partial U/\partial q_i)^{\circ}$ are considered to be positive at equilibrium, and $\beta > 0$ is some constant. If $\beta$ is taken sufficiently large, then $W$ will be positive-definite.

Indeed, $T$ is positive-definite with respect to the velocities and

$$W - T = \sum_{ij=\sigma+1}^{n} \gamma_{ij}' x_i x_j + \sum \gamma_{ij}' x_i x_j + \sum_{\substack{i\leqslant\sigma\\ i>n}} \beta_i x_i^2 + Y$$

where the second sum contains the terms of the second order which depend on the variations of the quasi-free coordinates, and $Y$ contains the terms of the higher order. The first sum is positive-definite by definition with respect to its variables; and an arbitrary diagonal minor $\Delta_k$ of the order $k > \sigma - n$ of the quadratic part of $W - T$ has the form

$$\Delta_k = \Delta_{n-\sigma}\beta^{k-n+\sigma} + C_{k,n-\sigma+1}\beta^{k-n+\sigma-1} + \dots$$

and will be positive for sufficiently large $\beta > 0$. Note also that, from the definition, all velocities of the quasi-free coordinates are linear in $v_1, \dots, v_\sigma$ and the time derivative of $W$ can be expressed as

$$\frac{dW}{dt} = \sum (Q_i + \mu_i) v_i - k_i N_i \sqrt{v_{ix}^2 + v_{iy}^2}$$

where all $\mu_i$ vanish at the equilibrium state. In a similar manner as it was done in the preceding section, it can be concluded that $dW/dt$ is a negative constant, and thus, the motion is stable.

Let $S$, the energy of the acceleration of the system, be represented as

$$S = S_2 + b_1\dot{v}_1 + \dots + b_n\ddot{q}_n + S_0$$

where $S_2$ is a part depending on the squares of the acceleration, $S_0$ is a part depending linearly on the accelerations; $b_1, \dots, b_n$ are vanishing at the equilibrium state. The equations of motion may be represented in the form

$$\frac{\partial S_2}{\partial \dot{v}_j} = -b_j + Q_j - \sum k_i N_i \frac{v_{ix}\beta_{ij}^1 + v_{iy}\beta_{ij}^2}{\sqrt{v_{ix}^2 + v_{iy}^2}}, \quad \frac{\partial S_2}{\partial \dot{q}_j} = -b_j + Q_j \quad (2.2)$$

If $\ddot{q}_{\sigma+1}, \dots, \ddot{q}_n$ are determined from the last $n - \sigma$ equations and substituted into the first $\sigma$ equations, we obtain

$$\frac{\partial S_2^*}{\partial \dot{v}_j} = -b_j' + Q_j - \sum k_i N_i \frac{v_{ix}\beta_{ij}^1 + v_{iy}\beta_{ij}^2}{\sqrt{v_{ix}^2 + v_{iy}^2}}$$

where $b_j'$ vanish at the equilibrium state and $S_2^*$ is $S_2$ where $\ddot{q}_{\sigma+1}, \dots, \ddot{q}_n$ are replaced by $\dot{v}_1, \dots, \dot{v}_\sigma$ from $\partial S_2/\partial \ddot{q}_j = 0$.

Indeed,

$$\frac{\partial S_2^*}{\partial \dot{v}_j} = \left(\frac{\partial S_2}{\partial \dot{v}_j}\right)^* + \sum_{i=\sigma+1}^{n} \frac{\partial S_2}{\partial \ddot{q}_i}\frac{\partial \ddot{p}_i}{\partial \dot{v}_j}$$

where $(\partial S_2/\partial \dot{v}_j)^*$ is $(\partial S_2/\partial \dot{v}_j)$ in which the last $n - \sigma$ relationships of (2.2) are taken into account. All terms on the right-hand side of the last equality which are independent of $\dot{v}_j$ will vanish at the origin of the coordinate system; and all terms depending linearly on $v_j$ will be such as though we put $\partial S_2/\partial \ddot{q}_j = 0$.

Since

$$S_2^* = \frac{1}{2} \sum_{ij=1}^{\sigma} a_{ij} \dot{v}_i \dot{v}_j$$

is a positive-definite function of $\dot{v}_1, \ldots, \dot{v}_\sigma$, then after multiplying (2.2) by $v_j$ and adding, we obtain

$$\frac{d}{dt} \frac{1}{2} \sum a_{ij} v_i v_j = -\frac{1}{2} \sum \frac{da_{ij}}{dt} v_i v_j + \sum (Q_j - b_j') v_j -$$

$$- \sum k_i N_i \sqrt{v_{ix}^2 + v_{iy}^2} = \sum (Q_j + \mu_j') v_j - \sum k_i N_i \sqrt{v_{ix}^2 + v_{iy}^2}$$

where all $\mu_j'$ vanish at the equilibrium state. Replacing the first part of the last equation by a larger value $-\theta \sqrt{(\Sigma a_{ij} v_i v_j)}$, as it was done in the first section, (this is permissible because of the stability of the motion (1.5) and (1.6)), we conclude, that the quasi-free coordinates vanish after some finite interval of time. Mechanically, this means that during the process of motion some small forces are acting on the components of the system corresponding to nonfree coordinates from the other components of the system. These forces, however, cannot move the nonfree part of the system away from the stagnation region.

This is apparently a manifestation of an essential difference between the dry and viscous friction. It is known that introducing partial dissipation by means of the forces of viscous friction results frequently in the asymptotic stability of equilibrium of a system at the minimum of potential energy, provided that there are no several equal natural frequencies of the system. The damping process is infinitely long, and the energy of the undamped members is transmitted to the damped members and thus is dissipated.

The dry friction forces, generally speaking, are capable of the dissipation only of a part of the energy of a system, moreover, the damping of the members with dry friction takes place during a finite period of time.

This is so because the dry friction forces are discontinuous functions of the velocity and remain undetermined for zero velocities; they may manifest themselves as the reactions of ideal constraints, since they do not work along any relative, admissible displacements of the contact points. The conversion of this theorem is not difficult. Indeed, the quasi-free coordinates, if the initial excitation did not disturb these coordinates nor their velocities, will remain constant, and motion of the system will be equivalent to a motion with additional constraints, at least in some neighborhood of the equilibrium state. If for an $\epsilon$-

neighborhood, entering in the determination of the equilibrium, we select an even smaller neighborhood, then the motion will be a motion with additional constraints for such a neighborhood.

Consequently, if the expansion of $U$ starts with a quadratic form which changes its sign, or a positive-definite form of the order $2m$, or it represents a form which changes the sign, then invoking the known theorems of Liapunov [7] and Chetaev [8], we come to the conclusion that the system is unstable in relation to the free-coordinates.

We also note that if the conclusion of the stability of the equilibrium would be obtained by imposing some other structural limitations on the forces, then the damping of the vibrations of nonfree coordinates will also take place within a finite period of time, since in the proof of this fact we used only the stability conditions.

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